A Discourse on Lifting Depth Indexes in λ-Calculus

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1. Overview

- Lifting, a.k.a. forward relocation, is a well-established operation on λ-terms where de Bruijn depth indexes express references to variables.
- We consider untyped λ-terms: $N, M ::= i \in \mathbb{N}^+ \mid @(N)M \mid \lambda M$, where we regard $\mathbb{N}^+$ as generated by the constructors 1 and S.
- The generic lift $M \mapsto \Uparrow^f M$ takes $f : \mathbb{N}^+ \to \mathbb{N}^+$ and operates thus:

$$\Uparrow^f i \stackrel{\text{def}}{=} f(i), \quad \Uparrow^f @(N)M \stackrel{\text{def}}{=} @(\Uparrow^f N)\Uparrow^f M, \quad \Uparrow^f \lambda M \stackrel{\text{def}}{=} \lambda \Uparrow^{(\mathord{\text{♀}} f)} M;$$

where we define $\mathord{\text{♀}} f$ by the clauses: $(\mathord{\text{♀}} f)(1) \stackrel{\text{def}}{=} 1$; $(\mathord{\text{♀}} f)(Si) \stackrel{\text{def}}{=} Sf(i)$.
- Usually $f \in \mathbf{B} \stackrel{\text{def}}{=} \{\tau(d,h) \mid d \in \mathbb{N} \text{ (depth)}, h \in \mathbb{N} \text{ (height)}\}$ where:

$$\tau(d,h) \stackrel{\text{def}}{=} i \mapsto \begin{cases} i & \text{if } i \le d \\ h+i & \text{if } i > d \end{cases}$$

- Notice the well-known fact $\mathord{\text{♀}}\tau(d,h) \stackrel{\text{ext}}{=} \tau(Sd,h)$ that follows easily.
- We set $\mathsf{id} \stackrel{\text{def}}{=} i \mapsto i$ (identity) and $\tau \stackrel{\text{def}}{=} \tau(0,1) \stackrel{\text{ext}}{=} i \mapsto Si$ (successor).

2. Overview (continued)

- We define composition in the usual way: $(f_2 \circ f_1)(i) \stackrel{\text{def}}{=} f_2(f_1(i))$.
- Notice: $\text{⫯}(f_2 \circ f_1) \stackrel{\text{ext}}{=} (\text{⫯} f_2) \circ (\text{⫯} f_1)$, $(\text{⫯} f) \circ \tau \stackrel{\text{ext}}{=} \tau \circ f$, $\text{⫯}\mathsf{id} \stackrel{\text{ext}}{=} \mathsf{id}$.
- The next well-known equations rule composition in the family $\mathbf{B}$:

$\tau(d_2, h_2) \circ \tau(d_1, h_1) \stackrel{\text{ext}}{=} \tau(d_1, h_2 + h_1)$ if $d_1 \leq d_2 \leq h_1 + d_1$;
$\tau(d_2, h_2) \circ \tau(d_1, h_1) \stackrel{\text{ext}}{=} \tau(h_2 + d_1, h_1) \circ \tau(d_2, h_2)$ if $d_2 \leq d_1$.

- The family $\mathbf{U} \stackrel{\text{def}}{=} \{\tau(0, h) \mid h \in \mathbb{N}\} \subset \mathbf{B}$ is closed under id, τ and $\circ$, not under ⫯; the family $\mathbf{B}$ is closed under id, τ and ⫯, but not under $\circ$.
- Two reasons for abstracting the function f in the definition of lift.

1. There are cases in which we must take the function f outside $\mathbf{B}$.
We will consider one of such cases in detail right after this slide.

2. The definition of lift only depends on the algebraic type of terms.
The theories of lift for different sets of terms can share a common base.

3. An example: strong normalization of $\lambda{\to}$

- Take strong normalization of $\lambda{\to}$ proved with Girard's candidates.

A pair $\langle \Gamma, M\rangle \in \mathcal{R}(A)$, the candidate at type A, satisfies $\Gamma \vdash M : A$.

- Moreover $\langle \Gamma, M\rangle \in \mathcal{R}(B \to A)$ iff $\langle \Delta, N\rangle \in \mathcal{R}(B)$ and $\Gamma \subseteq^{f} \Delta$

yield $\langle \Delta, @(N){\Uparrow}^{f} M\rangle \in \mathcal{R}(A)$, where f lifts M from Γ to Δ.

- The family in which f is taken must be closed under $\circ$ since A can

be an $\to$ type. No problem in the standard setting: we can take f in $\mathbf{U}$.

For $\langle \Gamma, M\rangle \in \mathcal{R}(B_2 \to B_1 \to A)$, $\langle \Delta_2, N_2\rangle \in \mathcal{R}(B_2)$ and $\langle \Delta_1, N_1\rangle \in \mathcal{R}(B_1)$ must imply

$\langle \Delta_2, @(N_2)@({\Uparrow}^{f_2} N_1){\Uparrow}^{f_2 \circ f_1} M\rangle \in \mathcal{R}(A)$, where f_1 lifts from Γ to Δ_1 and f_2 lifts from Δ_1 to Δ_2.

- If $\langle \Gamma, @(N)\lambda M\rangle \overset{\beta}{\to} \langle \Gamma.(N/1), M\rangle$, Tait's "ii" for $C \overset{\mathrm{def}}{=} B \to A$ is:

$\langle \Gamma.(N/1), @({\Uparrow}^{\tau}\overline{N})M\rangle \in \mathcal{R}(C) \Rightarrow \langle \Gamma, @(\overline{N})@(N)\lambda M\rangle \in \mathcal{R}(C)$.

$\langle \Delta, N_0\rangle \in \mathcal{R}(B) \Rightarrow \langle \Delta.({\Uparrow}^{f} N/1), {\Uparrow}^{\tau} N_0\rangle \in \mathcal{R}(B) \overset{H}{\Rightarrow} \langle \Delta.({\Uparrow}^{f} N/1), @({\Uparrow}^{\tau} N_0){\Uparrow}^{(\text{⫯} f)} @({\Uparrow}^{\tau}\overline{N})M\rangle \in$
$\mathcal{R}(A) \overset{IH}{\Rightarrow} \langle \Delta, @(N_0)@({\Uparrow}^{f}\overline{N})@({\Uparrow}^{f} N)\lambda{\Uparrow}^{(\text{⫯} f)} M\rangle \in \mathcal{R}(A) \Rightarrow \langle \Delta, @(N_0){\Uparrow}^{f} @(\overline{N})@(N)\lambda M\rangle \in \mathcal{R}(A)$

Here f lifts from Γ to Δ thus we need $(\text{⫯} f)$ to lift from $\Gamma.({\Uparrow}^{f} N/1)$ to $\Delta.({\Uparrow}^{f} N/1)$, and $(\text{⫯} f) \notin \mathbf{U}$.

4. Finite lifting

- Let $\mathbf{F}$ be the smallest family of functions containing $\mathbf{B}$ and closed under $\circ$. We can prove that $f \in \mathbf{F}$ iff the next properties hold:

1. strict monotonicity: $i_1 < i_2 \Rightarrow f(i_1) < f(i_2)$;
2. asymptotic uniformity: $(\exists l \in \mathbb{N})(\exists h \in \mathbb{N})\, i \geq l \Rightarrow f(i) = h + i$.

- The representation of $f \in \mathbf{F}$ with the list of its components in $\mathbf{B}$ is not unique, thus deciding $f_1 \stackrel{\text{ext}}{=} f_2$ is hard. Computing $\text{⫯} f$ is linear.
- A more convenient representation involves the stream of differences: $i \in \mathbb{N}^+ \mapsto \delta_i \stackrel{\text{def}}{=} f(i) - f(i-1) \in \mathbb{N}^+$ where we set δ_1 by $f(0) \stackrel{\text{def}}{=} 0$.
- By asymptotic uniformity: $i > l \Rightarrow \delta_i = 1$, so we can reduce the stream representing f to a list of length l that we shall denote by $[f]$.
- $[f]$ is unique up to trailing 1's, deciding $[f_1] \stackrel{\text{ext}}{=} [f_2]$ is linear in $l_1 \vee l_2$, computing application is linear, and $[\text{⫯} f] \stackrel{\text{ext}}{=} 1 :: [f]$ is constant.

5. Finite lifting (continued) and beyond

- Notice that $[\mathsf{id}] \overset{\text{ext}}{=} [\,]$ and that $[f](i)$ has the next properies:
  $[\,](i) = i; \qquad (\delta :: [f])(1) = \delta; \qquad (\delta :: [f])(Si) = \delta + [f](i).$
- Denoting with $\downarrow^{\delta}$ the tail operation iterated δ times, $\circ$ is such that:
  $[g] \circ [\,] \overset{\text{ext}}{=} [g]; \qquad [g] \circ (\delta :: [f]) \overset{\text{ext}}{=} [g](\delta) :: (\downarrow^{\delta}[g] \circ [f]).$
- To develop a formal theory of finite lifting we adapt it as follows:

1. We consider streams in place of lists: thus $[\,]$ never occurs and $f_1 \overset{\text{ext}}{=} f_2$ is $[f_1] \overset{\text{ext}}{=} [f_2]$. We work in $\mathbf{T}$, where uniformity is dropped.
2. We regard each $\delta \in \mathbb{N}^+$ as $S^{(\delta-1)}1$: thus $[f]$ becomes a stream of 1's and S's. Notice that the 1 follows the S's in the representation of δ.

- Then we may consider arbitrary streams of 1's and S's: the family $\mathbf{P}$.
- ? is a constructor of $\mathbf{P}$, *i.e.*, $[\,\text{?} f] \overset{\text{ext}}{=} 1 :: [f]$. The other constructor $\uparrow$, such that $[\uparrow f] \overset{\text{ext}}{=} S :: [f]$, comes from the position: $\uparrow f \overset{\text{def}}{=} \tau \circ f$.

6. An axiomatic presentation of lifting functions

- Our analysis highlighted two operations ⫯ and $\uparrow$ on lift functions that, when taken as primitive, serve as a base for a theory of lifting.

1. The lift functions are taken in the coinductive type $\mathbf{P}$ of streams with constructors ⫯ (push) and $\uparrow$ (next). $⫯\mathsf{id} \stackrel{\text{def}}{=} \mathsf{id} \stackrel{\text{def}}{=} ⫯^{\infty}$, $\uparrow\mathsf{top} \stackrel{\text{def}}{=} \mathsf{top} \stackrel{\text{def}}{=} \uparrow^{\infty}$.

2. Application takes a function and a coinductive positive integer:
$$1 \stackrel{\text{def}}{=} (⫯ f)(1); \qquad S f(i) \stackrel{\text{def}}{=} (⫯ f)(S i); \qquad S f(i) \stackrel{\text{def}}{=} (\uparrow f)(i).$$

3. Composition is defined coinductively by the next three clauses:
$$⫯(g \circ f) \stackrel{\text{def}}{=} (⫯ g) \circ (⫯ f); \ \uparrow(g \circ f) \stackrel{\text{def}}{=} (⫯ g) \circ (\uparrow f); \ \uparrow(g \circ f) \stackrel{\text{def}}{=} (\uparrow g) \circ f.$$

4. The condition $f \in \mathbf{T}$ corresponds to the requirement that f is total:
$$\mathsf{isinT}(f) \stackrel{\text{def}}{=} (\forall i \in \mathbb{N}^{+})\,(\exists j \in \mathbb{N}^{+})\, f(i) = j.$$

5. The condition $f \in \mathbf{F}$ corresponds to the inductive predicate $\mathsf{isinF}(f)$:
$$\mathsf{isinF}(\mathsf{id}); \qquad \mathsf{isinF}(f) \Rightarrow \mathsf{isinF}(⫯ :: f); \qquad \mathsf{isinF}(f) \Rightarrow \mathsf{isinF}(\uparrow :: f).$$

7. An axiomatic presentation of lifting functions (cont.)

6. We define the functions of the family $\mathbf{B}$ with: $\tau(d,h) \overset{\text{def}}{=} ⫯^{d}\uparrow^{h}⫯^{\infty}$.

7. The condition $f \in \mathbf{U}$ corresponds to the inductive predicate $\mathsf{isinU}(f)$:

$$\mathsf{isinU}(\mathsf{id}); \qquad \mathsf{isinU}(f) \Rightarrow \mathsf{isinU}(\uparrow :: f).$$

- We can prove that $(g \circ f)(i) \overset{\text{ext}}{=} g(f(i))$ and that the families $\mathbf{T}$, $\mathbf{F}$, $\mathbf{U}$ are closed under $\circ$ and contain id (as of $\mathbf{T}$, consider that $\mathsf{id}(i) \overset{\text{ext}}{=} i$).

- Injectivity of $\circ$, $\mathsf{isinT}(g) \vdash g \circ f_1 \overset{\text{ext}}{=} g \circ f_2 \Rightarrow f_1 \overset{\text{ext}}{=} f_2$, comes by coinduction on the ⫯'s of g and by induction on the $\uparrow$'s between them.

1. If $H(g)$ is the statement, by coind.: $(\forall g)\, g(1) = 1 \Rightarrow H(g)$.

2. By ind. on j: $(\forall g)\, g(1) = 1 \Rightarrow H(g) \vdash (\forall g)\, g(1) = j \Rightarrow H(g)$.

3. The former steps give $(\forall g)\, H(g)$ since $\mathsf{isinT}(g) \vdash (\exists j)\, g(1) = j$.

- Counterex. with $g \notin \mathbf{T}$: $(\forall f_1)(\forall f_2) \uparrow^{\infty} \circ f_1 \overset{\text{ext}}{=} \uparrow^{\infty} \circ f_2 \overset{\text{ext}}{=} \uparrow^{\infty}$.

- This theory is specified in Matita and it is used for lifting in $\lambda\delta$-2.

8. Subsets of free variable references

- We can use a lift function f to represent a subset $\{f\}$ of $\mathbb{N}^+$.

$\{f\} \stackrel{\text{def}}{=} (\mathbb{N}^+ - \mathsf{codom}\, f)$, that is to say: $i \in \{f\}$ iff $(\exists g)\, ↧^{i-1} f \stackrel{\text{ext}}{=} \uparrow g$.

- By so doing, we represent every finite subset of $\mathbb{N}^+$ with an $f \in \mathbf{F}$.

In particular $\varnothing \stackrel{\text{ext}}{=} \{\mathsf{id}\}$ and we can define notions like $\cup$ and $\subseteq$ in $\mathbf{P}$.

- Contrary to $\mathbf{T}$, see $(⫯\uparrow)^\infty \cup (\uparrow⫯)^\infty \stackrel{\text{ext}}{=} \uparrow^\infty$, $\mathbf{F}$ is closed under $\cup$.

- In $\mathbf{F}$ we can represent the free variable references of a term:

$\mathsf{fv}(i) \stackrel{\text{def}}{=} \tau(i{-}1, 1)$; $\mathsf{fv}(@(N)M) \stackrel{\text{def}}{=} \mathsf{fv}(N) \cup \mathsf{fv}(M)$; $\mathsf{fv}(\lambda M) \stackrel{\text{def}}{=} ↧\, \mathsf{fv}(M)$.

- Interestingly, $\mathsf{fv}(\Uparrow^g M) \stackrel{\text{ext}}{=} g \,\tilde{\circ}\, \mathsf{fv}(M)$ where $g \,\tilde{\circ}\, f$ (co-composition) is $\sim(g \circ \sim f)$ and $\sim f$ (complement) is f with $⫯$'s and $\uparrow$'s swapped.

- The complements in $\tilde{\circ}$ come from complementing $\mathsf{codom}\, f$ in $\{f\}$.

- $\mathbf{F}$ is not closed under $\sim$ but it is closed under $\tilde{\circ}$, that we define by:

$⫯(g \,\tilde{\circ}\, f) \stackrel{\text{def}}{=} (⫯g) \,\tilde{\circ}\, (⫯f)$; $\uparrow(g \,\tilde{\circ}\, f) \stackrel{\text{def}}{=} (⫯g) \,\tilde{\circ}\, (\uparrow f)$; $⫯(g \,\tilde{\circ}\, f) \stackrel{\text{def}}{=} (\uparrow g) \,\tilde{\circ}\, f$.

Thank you