Lebesgue's Dominated Convergence Theorem in Bishop's Style ⋆

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Abstract

We present a constructive proof in Bishop's style of Lebesgue's dominated convergence theorem in the abstract setting of ordered uniform spaces. The proof generalises to this setting a classical proof in the framework of uniform lattices presented by Hans Weber in "Uniform Lattices II: Order Continuity and Exhaustivity", in Annali di Matematica Pura ed Applicata (IV), Vol. CLXV (1993).

Key words: constructive mathematics, ordered sets, uniform spaces, Lebesgue's dominated convergence theorem, uniform lattices

⋆ Both authors have been partially supported by DAMA (Dimostrazione Assistita per la Matematica e l'Apprendimento), a strategic project of the University of Bologna.

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1 Introduction

Lebesgue's dominated convergence theorem represents an important milestone for the development of measure theory and probability theory. It provides sufficient conditions under which two limit processes, namely Lebesgue integration and pointwise convergence, commute. Classically, this theorem shows the superiority of the Lebesgue integral over the Riemann one for many theoretical purposes.

The most natural setting for this theorem is Lebesgue's integration theory of real or complex valued functions over a measure space. However, this is not the most general setting where the core of the theorem can be proved. Nor it is the most convenient setting to understand the role played by its side conditions.

A more satisfactory setting is that of topological Riesz spaces [5]. A Riesz space is an ordered vector space where the order relation is induced by a lattice structure. Integrable real valued functions over measure spaces form a Riesz space, which can be endowed with the topology induced by the integral norm. Since in a Riesz space functions are abstracted as points, pointwise convergence of sequences of function is abstracted by order convergence with respect to the order relation. Thus, in this setting, the theorem provides sufficient conditions under which two limit processes, namely the topological limit and order convergence, commute. The detailed proof of this fact can be found, for instance, in [5].

One important condition for the theorem to hold is *order continuity* that ties order convergence with topological convergence in the case of monotone sequences: $a_n \to a$ (according to the topology) whenever $a_n \uparrow a$ (a_n is a given increasing sequence whose supremum is a). In the setting of integrable real valued functions over a measure space, this condition corresponds to the statement of the Beppo Levi theorem, whose proof is quite deep and relies on the definition of Lebesgue's integral. Moreover, Beppo Levi's theorem fails for the Riemann integral. Thus we may claim that it is Beppo Levi's theorem, and not Lebesgue's, that shows the superiority of the Lebesgue integral over the Riemann integral. Indeed, once order continuity is assumed, the proof of Lebesgue's dominated convergence theorem becomes very easy and almost routine.

Hans Weber, during his studies on the generalisation of topological Riesz spaces and topological boolean rings [8], realized that even the setting of topological Riesz spaces is not the most general one where Lebesgue's dominated convergence theorem can be stated and proved. Indeed, in [9] he proves the theorem in the setting of uniform lattices, namely lattices endowed with a compatible uniform structure. With respect to a topological Riesz space, a uniform lattice need not be a vector space. Moreover, Weber refines the order continuity condition into simpler conditions (properties (σ) and exhaustivity), and superbly clarifies in this abstract setting where all conditions for the theorem on measure spaces come from (in particular the property of being dominated).

All the proofs and settings discussed so far are classical, but we are interested in obtaining a constructive proof of the theorem in Bishop's style in order to formalise it with an interactive theorem prover, as a first step for the formalisation of measure theory and probability theory. Moreover, we would like our proof to be stated in the most general setting where a constructive proof can be given.

Lebesgue's dominated convergence theorem has already been proved constructively by many authors. Chapter 6 of [2] is a thorough study of constructive integration theory, and it comprises a proof of the theorem in the setting of integration spaces. An integration space is a sort of measurable space where a measure has already been fixed in advance. This is constructively necessary since it is not constructively true that every measurable set can be measured by any measure. Since integration spaces (and even more measure spaces, introduced at the end of the chapter) are the best constructive counterparts of measure spaces, the proof of Bishop and Bridges is the counterpart to the classical proof in the setting of real valued functions on measure spaces. Instead, we are interested in a more general proof.

Spitters' PhD. thesis presents a study of integration theory in Bishop's style in the context of Riesz spaces. In [6,7] he proposes two different proofs of Lebesgue's dominated convergence theorem under slightly different assumptions. In particular, the proof in the second paper is especially valuable since it avoids the axiom of choice and any impredicative construction. In principle, these proofs should correspond to the classical proofs for Riesz spaces. However, we claim that what is called Lebesgue's dominated convergence theorem in Spitters' work is actually the proof of one of its corollaries, at least classically weaker than the theorem itself. Moreover, we think that the "spirit" of the classical proof is lost in Spitters' analysis: he still provides sufficient conditions under which two limit processes commute, but those are not topological convergence and order convergence. Instead, he considers convergence in norm and convergence in measure, thus avoiding any reference to order convergence and, consequently, to order continuity. The resulting proof is thus very different, both in spirit and technically, from Fremlin's one.

Spitters' proof cannot be adapted in the most general setting of uniform spaces since it is given for an archimedean lattice vector space that is also an algebra with a multiplicative unit. In fairness to him, it should be admitted that, at least in measure theory, all interesting models are Riesz spaces with a rich structure. Moreover, constructively an order relation often depends on a (pseudo-)metric or a uniformity. Finally, replacing convergence almost everywhere with convergence in measure cannot be avoided in point-free topology (see [7], Sect. 8).

In this paper we provide a constructive proof in Bishop's style of Lebesgue's dominated convergence theorem in the very general setting of ordered uniform spaces, i.e. ordered sets equipped with a compatible uniform space structure. Our proofs generalise their counterparts given by Weber, both in the sense that we weaken the underlying structure, and in the sense that we use only intuitionistic logic. They are more general than Fremlin's (and Spitters') ones since we only assume an ordered uniform space. Of course, such a drastic simplification of the underlying structure has been possible since we are only interested in the (constructive) analysis of Lebesgue's theorem, and not in a thorough theory of integration.

In Section 2 we recall the basic constructive theory (in Bishop's style) of partial orders (mainly inspired by [1]) and uniform spaces (also investigated in [3]). Section 3 is devoted to ordered uniform spaces and the proof of two versions of Lebesgue's dominated convergence theorem, respectively in the setting of uniform spaces (with property (σ)) whose restrictions to intervals are exhaustive, and in the setting of uniform spaces whose restrictions to intervals are order continuous. Neither version implies the other. Thus, in Section 5, we compare the properties (σ), order continuity and exhaustivity. Before that, in Section 4 we show that Weber's lattice uniformities are models of ordered uniform spaces.

2 Preliminaries

2.1 Logic

Our proofs are carried out in Bishop's style mathematics, that is standard mathematics developed with intuitionistic logic [2]. Moreover, we strive to avoid impredicative constructions. In particular, instead of working with uniformities defined as families of subsets satisfying the usual conditions, we prefer to work with set indexed bases, which is equivalent in the impredicative setting. Moreover, we restrict ourselves only to constructions that preserve set indexing, and we avoid axioms of choice.

We refrain from adopting a completely point-free approach by substituting formal basic entourages for set indexed basic entourages and a "forces" relation for membership. Moreover, we assume that the carriers of our structures form a set. However, in many interesting models we are interested in, the carrier is likely to be just a class. Thus, instantiating our results (as well as Spitters') to these models in a predicative setting may require additional work.

2.2 Ordered sets

Definition 2.1 (Ordered set) *An ordered set $(C, \nleq)$ is a data type C together with a propositional operation*[1] *$\nleq$ (called excess [1]) such that the following properties hold:*

(1) Co-reflexivity: $\forall x : C.\neg(x \nleq x)$
(2) Co-transitivity: $\forall x, y, z : C.x \nleq y \Rightarrow x \nleq z \vee z \nleq y$

As in the classical case, if $\nleq$ is an excess operation, the same holds for $\nleq^{-1}$. This allows to omit dualized definitions and statements in the sequel.

Definition 2.2 (Apartness, equality, less or equal) *Let $(C, \nleq)$ be an ordered set.*

(1) $x \neq y$ iff $x \nleq y \vee y \nleq x$.
(2) $x = y$ iff $\neg(x \neq y)$.
(3) $x \leq y$ iff $\neg(x \nleq y)$.

$(C, \neq)$ endowed with the equality relation induced by $\nleq$ is a set in Bishop's terminology. Moreover, the excess and less or equal propositional operations are relations w.r.t. the equality. From the co-reflexivity and co-transitivity properties of $\nleq$ it immediately follows reflexivity and transitivity of $\leq$ and $=$, and co-reflexivity and co-transitivity of $\neq$.

Lemma 2.3 *Let $(C, \nleq)$ be an ordered set and $a, b, a', b' \in C$ such that $a \nleq b$, $a \leq a'$, $b' \leq b$. Then $a' \nleq b'$.*

Proof. By co-transitivity applied to the hypothesis $a \nleq b$ we have $a \nleq a' \vee a' \nleq b$. Since $a \leq a'$ by hypothesis, we have $a' \nleq b$. By co-transitivity once more, we have $a' \nleq b'$ or $b' \nleq b$. The latter cannot be since $b' \leq b$. □

Definition 2.4 (Strong supremum) *Let $(C, \nleq)$ be an ordered set and (a_i) a sequence in C. $a \in C$ is a strong supremum of (a_i) if $\forall i \in \mathbb{N}.a_i \leq a$ and*

[1] We call C a *data type* and not a set since we will ignore its equality. Correspondingly, we require $\nleq$ to be only a propositional operation, in Bishop's sense, and not a relation, since we are not interested in the preservation of any equivalence relation on C. Any ordered set will turn out to be a set with an excess relation when we will induce an equality on C starting from the excess propositional operation.

$\forall b \in C. a \not\leq b \Rightarrow \exists i \in \mathbb{N}. a_i \not\leq b.$

This definition is a restriction to sequences of Baroni's definition of strong supremum [1]. This suffices for our aim and it also simplifies the quest for models.

A strong supremum, when it exists, is unique. More than this, a strong supremum is necessary a weak supremum, i.e. the least upper bound of the sequence:

Fact 2.5 *Let $(C, \not\leq)$ be an ordered set, (a_i) a sequence in C and $a \in C$ the strong supremum of (a_i). Then for all $b \in C$ such that $a_i \leq b$ for all $i \in \mathbb{N}$, we have $a \leq b$.*

We write $a_i \uparrow a$ when (a_i) is an increasing sequence, whose strong supremum is a.

Lemma 2.6 *Let $(C, \not\leq)$ be an ordered set and (m_n) a strictly increasing sequence of natural numbers. If a and (a_n) are in C and $a_n \uparrow a$, then $a_{m_n} \uparrow a$.*

Proof. Obviously, for all n we have $a_{m_n} \leq a$ since $a_i \leq a$ for any i. We need to prove $\forall b \in C. a \not\leq b \Rightarrow \exists i \in \mathbb{N}. a_{m_i} \not\leq b$. Fix $b \in C$ such that $a \not\leq b$. Since $a_n \uparrow a$, by definition of strong supremum there exists $\bar{n} \in \mathbb{N}$ such that $a_{\bar{n}} \not\leq b$. Since (a_n) is increasing, $\forall i \geq \bar{n}. a_{\bar{n}} \leq a_i$. By Lemma 2.3, $\forall i \geq \bar{n}. a_i \not\leq b$. Since (m_n) is a strictly increasing sequence of natural numbers, it is easy to prove by induction that $m_{\bar{n}} \geq \bar{n}$ and therefore $a_{m_{\bar{n}}} \not\leq b$. $\square$

Definition 2.7 (Order convergence) *Let $(C, \not\leq)$ be an ordered set and a and (a_i) in C. We say that (a_i) order converges to a (written $a_i \xrightarrow{o} a$) iff there exist an increasing sequence (l_i) and a decreasing sequence (u_i) in C such that $l_i \uparrow a$ and $u_i \downarrow a$ and for all $i \in \mathbb{N}$ the strong infimum of $(a_{i+n})_{n \in \mathbb{N}}$ is l_i and the strong supremum is u_i.*

Definition 2.8 (Segment) *Let $(C, \not\leq)$ be an ordered set and $a, b \in C$. The segment $[a, b]$ is the set $\{x | a \leq x$ and $x \leq b\}$.*

Clearly, the restriction of an ordered set to a segment is itself canonically endowed with an order structure. Moreover, the following lemma shows that strong suprema are preserved.

Lemma 2.9 *Let $(C, \not\leq)$ be an ordered set, $l, u \in C$ and (a_i) and a in $C \cap [l, u]$. If $a_i \uparrow a$ in C, then $a_i \uparrow a$ in $C \cap [l, u]$.*

Proof. Obviously (a_i) is increasing and a is an upper bound of a also in $C \cap [l, u]$. Let $b \in C \cap [l, u]$ such that $a \not\leq b$ in $C \cap [l, u]$. Then $a \not\leq b$ also in C and by definition of strong supremum in C there exists $i \in \mathbb{N}$ such that $a_i \not\leq b$ in C. So it does also in $C \cap [l, u]$. $\square$

Definition 2.10 (Convex set) *Let $(C, \nleq)$ be an ordered set. We say that a set $U \subseteq C \times C$ is convex iff $\forall (a,b) \in U.a \leq b \Rightarrow [a,b]^2 \subseteq U$.*

Our definition of convex set is a slight restriction of the usual one (see, for instance, [8]) on the cartesian product $C \times C$ (endowed with the product order).

The following principle of upper locatedness for sequences always holds classically.

Definition 2.11 (Upper locatedness) *Let $(C, \nleq)$ be an ordered set. The sequence (a_i) is upper located [1] if $\forall x, y \in C.y \nleq x \Rightarrow (\exists i \in \mathbb{N}.a_i \nleq x) \vee (\exists b \in C.y \nleq b \wedge \forall i \in \mathbb{N}.a_i \leq b)$.*

Lemma 2.12 *Let $(C, \nleq)$ be an ordered set and (a_i) and a in C such that $a_i \uparrow a$. Then (a_i) is upper located in C.*

Proof. Fix $x, y \in C$ such that $y \nleq x$. We need to prove $(\exists i \in \mathbb{N}.a_i \nleq x) \vee (\exists b \in C.y \lneq b \wedge \forall i \in \mathbb{N}.a_i \leq b)$. By co-transitivity, either $y \nleq a$ or $a \nleq x$. In the first case, we prove the right hand side of the thesis by taking a for b. In the second case, by definition of strong supremum, there exists $i \in \mathbb{N}$ such that $a_i \nleq x$. □

2.3 Uniform spaces

Definition 2.13 (Uniform space) *A uniform space $(C, \neq, \Phi)$ is a set $(C, \neq)$ equipped with an inhabited family Φ (called uniformity base) of subsets of the cartesian product $C \times C$ (called basic entourages) with the following properties:*

(1) $\forall U \in \Phi.\{(x,y) | \neg(x \neq y) \in C\} \subseteq U$
(2) $\forall U, V \in \Phi.\exists W \in \Phi.W \subseteq U \cap V$
(3) $\forall U \in \Phi.\exists V \in \Phi.V \circ V \subseteq U$
(4) $\forall U \in \Phi.U = U^{-1}$

Some authors do not require entourages to be symmetric, replacing property (4) above with the following: $\forall U \in \Phi.U^{-1} \in \Phi$. Our choice allows some technical simplifications and is adopted, for instance, by Engelking in [4].

The usual definition of uniform spaces is in terms of (not necessarily basic) entourages. An entourage is any superset of some basic entourage. We do not follow this approach since the family of all entourages is necessarily a proper class in an impredicative setting. Indeed, the class Φ of all entourages is closed w.r.t. the following property: $\forall U \in \Phi.\forall V \in 2^{C \times C}.U \subseteq V \Rightarrow V \in \Phi$ where the quantification of V is on the powerset of the $C\times$. On the contrary, to work in a predicative setting it is sufficient to assume that the class Φ of all basic entourages is set indexed and that all quantifications in the definition of Φ are on the set of indexes. In what follows, we will tacitly assume this.

In [3], Bridges and Vîţă introduce a constructive version of uniform spaces that adds to the usual definition the new condition $\forall U \in \Phi.\exists V \in \Phi.\forall x \in C \times C.(x \in U \vee x \notin V)$, always classically valid. The condition is not required here.

Definition 2.14 (Cauchy sequence) *A sequence (a_i) of points of a uniform space $(C, \neq, \Phi)$ is Cauchy iff $\forall U \in \Phi.\exists n \in \mathbb{N}.\forall i, j \geq n.(a_i, a_j) \in U$.*

Definition 2.15 (Uniform convergence) *A sequence (a_i) of points of a uniform space $(C, \neq, \Phi)$ converges to a point $a \in C$ (written $a_i \to a$) if $\forall U \in \Phi.\exists n \in \mathbb{N}.\forall i \geq n.(a, a_i) \in U$.*

Lemma 2.16 *Let $(C, \neq, \Phi)$ be a uniform space and (a_i) and a in C such that $a_i \to a$. Then (a_i) is Cauchy.*

Proof. Fix $U \in \Phi$. We need to prove $\exists n \in \mathbb{N}.\forall i, j \geq n.(a_i, a_j) \in U$. By property (3) of a uniform space, there exists $V \in \Phi$ such that $V \circ V \subseteq U$. By Definition 2.15, there exists $n \in \mathbb{N}$ such that $\forall i \geq n.(a_i, a) \in V$. Thus $\forall i, j \geq n.(a_i, a_j) \in V \circ V^{-1} = V \circ V \subseteq U$. □

An uniform space $(C, \neq, \Phi)$ is complete if every Cauchy sequence in C converges to a point in C.

Definition 2.17 (Restricted uniformity) *Let $(C, \neq, \Phi)$ be a uniform space and X a subset of C. We call the family $\{U \cap X \times X | U \in \Phi\}$ the restricted uniformity base on X.*

The definition is well posed, as the properties listed in Definition 2.13 hold.

Fact 2.18 *Let $(C, \neq \Phi)$ be a uniform space, X a subset of C and (a_i) in X. If (a_i) is Cauchy in X, then (a_i) is Cauchy in C.*

3 Ordered uniform spaces and Lebesgue's dominated convergence theorem

3.1 Ordered uniform spaces

Definition 3.1 (Ordered uniform space) *A triple $(C, \nleq, \Phi)$ is an ordered uniform space iff $(C, \nleq)$ is an ordered set, (C, Φ) is a uniform space and every* *basic entourage $U \in \Phi$ is convex.*

Lemma 3.2 *Let $(C, \nleq, \Phi)$ be an ordered uniform space and $l, u \in C$. Let (a_i) and a in $C \cap [l, u]$. If $a_i \to a$ in C then $a_i \to a$ in $C \cap [l, u]$.*

Proof. By Definition 2.15, $\forall U \in \Phi. \exists m \in \mathbb{N}. \forall i \geq m.(a_i, a) \in U$. Since $a_i, a \in C \cap [l, u]$ for each $i \in \mathbb{N}$, the pair $(a_i, a) \in U \cap [l, u]^2$. Thus $\forall U \in \Phi. \exists m \in \mathbb{N}. \forall i \geq m.(a_i, a) \in U \cap [l, u]^2$. $\square$

Theorem 3.3 (Sandwich) *Let $(C, \nleq, \Phi)$ be an ordered uniform space. Let $l \in C$ and $(a_i), (x_i), (b_i)$ be sequences in C such that $\forall i \in \mathbb{N}.a_i \leq x_i \leq b_i$ and $a_i \to l$ and $b_i \to l$. Then $x_i \to l$.*

Proof. We need to prove $\forall U \in \Phi. \exists m \in \mathbb{N}. \forall i \geq m.(x_i, l) \in U$. Fix $U \in \Phi$ and let $V \in \Phi$ such that $V \circ V \subseteq U$. Let $W \in \Phi$ such that $W \circ W \subseteq V$. Thus $\exists m \in \mathbb{N}. \forall i \geq m.(a_i, l) \in W \wedge (b_i, l) \in W$. Therefore $\exists m \in \mathbb{N}. \forall i \geq m.(a_i, b_i) \in V$. Since V is convex, $\exists m \in \mathbb{N}. \forall i \geq m.[a_i, b_i]^2 \subseteq V$. Hence $\exists m \in \mathbb{N}. \forall i \geq m.(x_i, a_i) \in V \wedge (a_i, l) \in W \subseteq V$. Thus $\exists m \in \mathbb{N}. \forall i \geq m.(x_i, l) \in V \circ V \subseteq U$. $\square$

Definition 3.4 (Order continuity) *Let the triple $(C, \nleq, \Phi)$ be an ordered uniform space. We say that the uniformity is order continuous iff for all (a_i) and a in C, $a_i \uparrow a \Rightarrow a_i \to a$ and $a_i \downarrow a \Rightarrow a_i \to a$.*

Order continuity is a very natural requirement since it tightens the connection between the order and uniform structures in ordered uniform spaces. In [8,9], Weber shows that order continuity is better understood as a consequence of the combination of properties (σ) and exhaustivity, to be discussed in the following sections.

3.2 Uniformities with property (σ)

Definition 3.5 (Property (σ)) *Let $(C, \nleq, \Phi)$ be an ordered uniform space. The uniformity satisfies property (σ) iff $\forall U \in \Phi. \exists (U_n). \forall (a_n). \forall a.a_n \uparrow a \Rightarrow (\forall n. \forall i, j \geq n.(a_i, a_j) \in U_n) \Rightarrow (a_1, a) \in U$.*

Classically, for l-groups, the uniformity induced by a Riesz pseudonorm satisfies (σ) iff the pseudonorm is σ-subadditive ([8], Proposition 3.16). Similarly, the uniformity induced by a submeasure on a boolean ring satisfies (σ) iff the submeasure is σ-subadditive ([8], Proposition 3.17). Thus property (σ) captures the σ-additivity of measure spaces in a way that is more faithful than order continuity.

Lemma 3.6 *Let $(C, \nleq, \Phi)$ be an ordered uniform space with property (σ). Suppose (a_i), a in C such that $a_i \uparrow a$. If (a_i) is Cauchy, then $a_i \to a$.*

Proof. Fix $U \in \Phi$. We need to prove $\exists m \in \mathbb{N}.\forall i \geq m.(a_i, a) \in U$. Let (U_n) as in Definition 3.5 and let (m_n) in $\mathbb{N}$ be the sequence defined by recursion as follows. For the base case, since (a_i) is Cauchy, there exists $k \in \mathbb{N}$ such that $\forall j, j' \geq k.(a_j, a_{j'}) \in U_0$; take k for m_0. For the inductive case, since (a_i) is Cauchy, there exists $k \in \mathbb{N}$ such that $\forall j, j' \geq k.(a_j, a_{j'}) \in U_{n+1}$. Take $\max\{k, m_n + 1\}$ for m_{n+1}. The sequence (m_n) is strictly increasing by construction. Thus $a_{m_n} \uparrow a$ by Lemma 2.6. Thus, by property (σ), $(a_{m_1}, a) \in U$. Take m_1 and let $i \geq m_1$. Since (a_{m_n}) is increasing, $a_i \in [a_{m_1}, a]$. Since U is convex and $(a_{m_1}, a) \in U$, also $(a_i, a) \in U$. □

It should be noted that the property (σ) is not hereditary, in the sense that it is not preserved under restrictions, even to closed intervals. This is a consequence of the fact that, even classically, a (strong) supremum in an ordered set restricted to a segment is not necessarily a (strong) supremum in the whole set.

3.3 Exhaustive order uniformities

Lemma 3.6 is not sufficient to grant that an ordered uniform space with property (σ) is also order continuous. Classically, we also need *exhaustivity*: when restricted to sequences, this is precisely the condition ensuring that any monotone sequence is Cauchy.

Constructively, the classical definition of exhaustivity does not admit interesting models. For instance, consider the unit interval $[0, 1]$ endowed with the usual complete uniformity and order structure. Classically, its order uniformity is exhaustive. Constructively, this does not hold since it is not true that any monotone sequence in $[0, 1]$ has a least upper bound (otherwise: the sequence would be Cauchy by exhaustivity; so it would converge by metric completeness to some limit; finally, this limit would be a supremum since for any increasing sequence (a_n) of real numbers, $a_n \to a$ implies $a_n \uparrow a$).

We replace the classical definition of exhaustivity with the following one, which is classically equivalent.

Definition 3.7 (Exhaustivity) *The uniformity Φ of the ordered uniform space $(C, \not\leq, \Phi)$ is exhaustive if any increasing sequence that is upper located, and any decreasing sequence that is lower located, is Cauchy.*

For instance, Banach lattices are constructive models of exhaustive uniformities.

Usually we are interested in subspaces of a given ordered uniform space that are endowed with an exhaustive uniformity. The following theorem provides in this case sufficient conditions for a "local" version of order continuity.

Lemma 3.8 *Let $(C, \not\leq, \Phi)$ be an ordered uniform space with property (σ). Let $l, u \in C$ such that the uniformity induced on $C \cap [l, u]$ is exhaustive. If (a_i) is a sequence in $C \cap [l, u]$ and a a point in C such that $a_i \uparrow_C a$, then $a \in [l, u]$ and $a_i \to a$ in $C \cap [l, u]$.*

Proof. To prove $a \in [l, u]$, it suffices to notice that $l \leq a_1 \leq a$ and $a \leq u$ by Fact 2.5. Thus $a_i \uparrow a$ in $C \cap [l, u]$ by Lemma 2.9 and (a_i) is upper located in $C \cap [l, u]$ by Lemma 2.12. By exhaustivity of the uniformity restricted to $C \cap [l, u]$, (a_i) is Cauchy w.r.t. $C \cap [l, u]$. By Fact 2.18, (a_i) is Cauchy also w.r.t. C. Thus, by Theorem 3.6, $a_i \to_C a$. Finally, by Lemma 3.2, we conclude $a_i \to a$ in $C \cap [l, u]$. □

3.4 Lebesgue's dominated convergence theorem

We present two versions of Lebesgue's dominated convergence theorem. The first deals with ordered uniform spaces (with property (σ)) whose restrictions to intervals are exhaustive. The second deals with ordered uniform spaces whose restrictions to intervals are order continuous. Even in spite of the fact that order continuity is implied by property (σ) and exhaustivity (Theorem 5.1), neither version implies the other. This is a consequence of property (σ) not being hereditary, as already observed.

Theorem 3.9 (Lebesgue) *Let $(C, \not\leq, \Phi)$ be an ordered uniform space with property (σ) and such that, for all $l, u \in C$, the uniformity induced on $C \cap [l, u]$ is exhaustive. Let (a_i) be a sequence in C and $l, u \in C$ such that $\forall i \in \mathbb{N}. a_i \in C \cap [l, u]$. Finally, let a be a point in C such that $a_i \xrightarrow{o} a$ in C. Then $a \in C \cap [l, u]$ and $a_i \to a$ in $C \cap [l, u]$.*

Proof. The uniformity induced on $C \cap [l, u]$ is exhaustive by hypothesis. From $a_i \xrightarrow{o} a$ in C, there exist (x_i) and (y_i) such that $x_i \uparrow a$ and $y_i \downarrow a$ and for all $i \in \mathbb{N}$, $x_i \leq a_i \leq y_i$. Thus, by Lemma 2.12, (x_i) is upper located and (y_i) is lower located. By Lemma 3.8 we have $a \in C \cap [l, u]$ and $x_i \to a$ in $C \cap [l, u]$ and $y_i \to a$ in $C \cap [l, u]$. Since $\forall i \in \mathbb{N}. x_i \leq a_i \leq y_i$, by Theorem 3.3 we have $a_i \to a$ in $C \cap [l, u]$. □

Theorem 3.10 (Lebesgue) *Let $(C, \not\leq, \Phi)$ be an ordered uniform space such that for all $l, u \in C$ the uniformity induced on $C \cap [l, u]$ is order continuous. Let (a_i) be a sequence in C and $l, u \in C$ such that $\forall i \in \mathbb{N}. a_i \in C \cap [l, u]$. Finally, let a be a point in C such that $a_i \xrightarrow{o} a$ in C. Then $a \in C \cap [l, u]$ and $a_i \to a$ in $C \cap [l, u]$.*

Proof. The uniformity induced on $C \cap [l,u]$ is order continuous by hypothesis. From $a_i \xrightarrow{o} a$ in C, there exist (x_i) and (y_i) such that $x_i \uparrow a$ and $y_i \downarrow a$ and for all $i \in \mathbb{N}$, $x_i \leq a_i \leq y_i$. Thus, by definition of order continuity, $x_i \to a$ in $C \cap [l,u]$ and $y_i \to a$ in $C \cap [l,u]$. To prove $a \in [l,u]$, it suffices to notice that $l \leq a_1 \leq a$ and that $a \leq u$ by Fact 2.5. Finally, since $\forall i \in \mathbb{N}. x_i \leq a_i \leq y_i$, by Theorem 3.3, $a_i \to a$ in $C \cap [l,u]$. $\square$

4 Ordered uniform spaces induced by lattice uniformities

4.1 Lattices

Definition 4.1 (Lattice) *A lattice is a tuple $(C, \neq, \vee, \wedge)$ where $(C, \neq)$ is a set and $\vee$ and $\wedge$ are strongly extensional functions of type $C \to C \to C$ s.t.:*

(1) $\vee, \wedge$ are idempotent, commutative and associative
(2) $\vee, \wedge$ are absorbent

Definition 4.2 (Induced excess relation) *Let $(C, \neq, \vee, \wedge)$ be a lattice. We write $x \not\leq y$ for $x \neq x \wedge y$.*

Fact 4.3 *Let $(C, \neq, \vee, \wedge)$ be a lattice. Then $(C, \not\leq)$ is an ordered set. Moreover, the apartness induced by $\not\leq$ is $\neq$.*

Fact 4.4 *Let $(C, \neq, \vee, land)$ be a lattice. For all $a, b, c, d \in C$:*

(1) $a \leq b$ and $a \leq c$ imply $a \leq b \wedge c$
(2) $b \leq a$ and $c \leq a$ imply $b \vee c \leq a$

Fact 4.5 *Let $(C, \neq, \vee, \wedge)$ be a lattice. For $a, b \in C$ such that $a \leq b$, $a \wedge b = a$.*

Definition 4.6 ($\tilde{U}$) *Let $(C, \neq, \vee, \wedge)$ be a lattice and let $U \subseteq C \times C$. We define $\tilde{U}$ as $\{(a,b) \in C \times C | [a \wedge b, a \vee b]^2 \subseteq U\}$.*

When $\mathcal{S}$ is a set indexed family of subsets of $C \times C$, the family $\{\tilde{U} | U \in \mathcal{S}\}$ is also set indexed.

Lemma 4.7 *Let $(C, \neq, \vee, \wedge)$ be a lattice and let $U \subseteq C \times C$. Then $\tilde{U} \subseteq U$ and $\tilde{U}$ is convex.*

Proof. Let $(a,b) \in \tilde{U}$. Since $a \wedge b \leq a \leq a \vee b$ and $a \wedge b \leq b \leq a \vee b$, both a and b are in $[a \wedge b, a \vee b]$. By definition of $\tilde{U}$, $(a,b) \in U$ and thus $\tilde{U} \subseteq U$.

Let $(a,b) \in \tilde{U}$ such that $a \leq b$. We need to prove $[a,b]^2 \subseteq \tilde{U}$. Let $(a',b') \in [a,b]^2$. Thus $a \leq a' \leq b$ and $a \leq b' \leq b$. We need to prove $[a' \wedge b', a' \vee b']^2 \subseteq U$.

Take $(c,d) \in [a' \wedge b', a' \vee b']^2$. Thus $a' \wedge b' \leq c \leq a' \vee b'$ and $a' \wedge b' \leq d \leq a' \vee b'$. We need to prove $(c,d) \in U$. From $a \leq a'$, $a \leq b'$, $a' \leq b$, $b' \leq b$ and Fact 4.4 we have $a \wedge b \leq a \leq a' \wedge b' \leq c \leq a' \vee b' \leq b \leq a \vee b$ and $a \wedge b \leq a \leq a' \wedge b' \leq d \leq a' \vee b' \vee b \wedge a \vee b$. By definition of $\tilde{U}$ we have $(c,d) \in U$. □

4.2 Product uniform spaces and uniformly continuous functions

Definition 4.8 (Uniform continuity) *A function f from a uniform space $(C, \neq, \Phi)$ to a uniform space $(C', \neq', \Phi')$ is uniformly continuous if $\forall U \in \Phi'.\exists V \in \Phi.V \subseteq f^{-1}(U)$.*

Fact 4.9 *Any composition of uniformly continuous functions is uniformly continuous.*

Definition 4.10 (Product uniform space) *Suppose we are given two uniform spaces $(C_1, \neq_1, \Phi_1)$ and $(C_2, \neq_2, \Phi_2)$.*

Let $\neq$ the relation defined on $C_1 \times C_2$ by $(a_1, a_2) \neq (b_1, b_2)$ iff $a_1 \neq b_1$ or $a_2 \neq b_2$.

Let Φ be the family of subsets of $(C_1 \times C_2)^2$ defined by $U \in \Phi$ iff there exist $U_1 \in \Phi_1$ and $U_2 \in \Phi_2$ such that for all $a_1, b_1 \in C_1$ and for all $a_2, b_2 \in C_2$

$$((a_1, a_2), (b_1, b_2)) \in U \iff (a_1, b_1) \in U_1 \wedge (a_2, b_2) \in U_2$$

We call $(C_1 \times C_2, \neq, \Phi)$ the product uniform space.

The previous definition is well posed in the sense that the triple $(C_1 \times C_2, \neq, \Phi)$ is a uniform space in the sense of Definition 2.13. Moreover, when the families of basic entourages Φ_1 and Φ_2 are set indexed, the family Φ is also set indexed.

Lemma 4.11 *Let $(C, \neq, \Phi)$ be a uniform space and U a basic entourage of the product uniform space $C \times C$. Then there exists a basic entourage $V \in \Phi$ s.t. $\{(a_1, a_2), (b_1, b_2) | (a_1, b_1) \in V \wedge (a_2, b_2) \in V\} \subseteq U$.*

Proof. Let U be a basic entourage of the product uniform space $C \times C$. By definition of product uniform space, there exists $V_1, V_2 \in \Phi$ such that $U = \{(a_1, a_2), (b_1, b_2) | (a_1, b_1) \in V_1 \wedge (a_2, b_2) \in V_2\}$. By property (2) of a uniform space base, there exists $V \in \Phi$ such that $V \subseteq V_1 \cap V_2$. □

Definition 4.12 (Uniform lattice) *A uniform lattice* $(C, \neq, \vee, \wedge, \Phi)$ *is a lattice* $(C, \neq, \vee, \wedge)$ *such that* $(C, \neq, \Phi)$ *is a uniform space and* $\vee, \wedge$ *are uniformly continuous.*

Theorem 4.13 (Existence of a convex base) *Let* $(C, \neq, \vee, \wedge, \Phi)$ *be a uniform lattice.* $\forall U \in \Phi . \exists V \in \Phi . V \subseteq \tilde{U} \subseteq U$.

Proof. $\tilde{U}$ is convex by Lemma 4.7. By property (3) of a uniform space base, let $W \in \Phi$ such that $W \circ W \subseteq U$. Consider the uniformly continuous function $f : C \times C \times C \to C$ (where the product $C \times C \times C$ is endowed with the product uniformity) defined as $f(x_1, x_2, x_3) = (x_1 \wedge (x_2 \vee x_3)) \vee (x_2 \wedge x_3)$. The function f is uniformly continuous being a composition of uniformly continuous functions (Fact 4.9). By definition of uniform continuity and by Lemma 4.11, there exists $V \in \Phi$ such that $\forall ((x_1, x_1'), (x_2, x_2'), (x_3, x_3')) \in V^3 . (f(x_1, x_2, x_3), f(x_1', x_2', x_3')) \in W$. We prove $V \subseteq \tilde{U}$. Let $(a, b) \in V$ and $(x, y) \in [a \wedge b, a \vee b]^2$. Since $f(x, a, b) = x$ and $f(x, a, a) = a$ by Fact 4.5, we have $(x, a) \in W$, and similarly $(y, a) \in W$. Thus $(x, y) \in W \circ W^{-1} = W \circ W \subseteq U$. We conclude $(a, b) \in \tilde{U}$, and thus $V \subseteq \tilde{U}$. $\square$

When the family Φ of basic entourages is set indexed, the family $\{\tilde{U} | U \in \Phi\}$ is also set indexed. In view of the previous theorem, we have thus proved, constructively and predicatively, that any uniform space with a set indexed base admits an equivalent set indexed base formed by convex basic entourages.

5 Order continuity, exhaustivity and property (σ)

In this section we show that the classical relations between order continuity, exhaustivity and property (σ) also hold constructively.

Theorem 5.1 *If the uniformity* Φ *of an ordered uniform space* $(C, \nleq, \Phi)$ *is exhaustive and satisfies* (σ)*, then it is order continuous.*

Proof. Assume (a_i) and a in C such that $a_i \uparrow a$. By Lemma 2.12, (a_i) is upper located. Thus, by exhaustivity, (a_i) is Cauchy and so $a_i \to a$ by Theorem 3.6. $\square$

Theorem 5.2 *If an ordered uniform space* $(C, \nleq, \Phi)$ *is order continuous, then it satisfies* (σ).

Proof. Fix $U \in \Phi$ and let $V \in \Phi$ such that $V \circ V \subseteq U$. Take $U_n = V$ for each $n \in \mathbb{N}$. Now consider (a_i) and a in C such that $a_i \uparrow a$ and suppose

$\forall n.\forall i, j \geq n.(a_i, a_j) \in U_n$. In particular, $\forall i, j.(a_i, a_j) \in U_0 = V$. We need to prove $(a_1, a) \in U$. By order continuity and the hypothesis $a_i \uparrow a$ we have $a_i \to a$. Thus $\forall n.\exists m.\forall i \geq m.(a_i, a) \in U_n = V$. Let m such that $\forall i \geq m.(a_i, a) \in V$. Then $(a_m, a) \in V$ and $(a_1, a_m) \in V$. Thus $(a_1, a) \in V \circ V \subseteq U$. $\square$

Exhaustivity and property (σ) are sufficient, but not necessary, conditions for order continuity. Indeed, order continuity fails to imply exhaustivity as the following counter-example shows, even classically.

Example 5.3 *Consider the real numbers with the complete uniformity induced by the usual metric and order structures. By definition, $a \nleq b$ iff $|a-b| = a - b > 0$ and order continuity holds. Consider now the monotone sequence $(i)_{i\in\mathbb{N}}$. The sequence is upper located: let $x, y \in \mathbb{R}$ such that $y \nleq x$; since the reals are Archimedean, there exists $n \in \mathbb{N}$ such that $i_n = n \nleq x$. The sequence is not Cauchy since it diverges. Thus, the real numbers uniformity is not exhaustive.*

Order completeness coincides with exhaustivity together with property (σ) under the additional hypothesis of order completeness.

Definition 5.4 (Order completeness) *The ordered set $(C, \nleq)$ is order complete iff all upper located sequences have a strong supremum and all lower located sequences have a strong infimum.*

Theorem 5.5 *If an order complete ordered uniform space $(C, \nleq, \Phi)$ is order continuous, then its uniformity is exhaustive.*

Proof. Let (a_i) be an increasing sequence in C that is upper located. By order completeness, there exists $a \in C$ such that $(a_i) \uparrow a$. By order continuity, $a_i \to a$. By Lemma 2.16, (a_i) is Cauchy. Since (a_i) was chosen arbitrarily, the order uniformity is exhaustive. $\square$.

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